# 5 Counting

## 5.1 The Basics of Counting

1. product rule (used when a procedure is made up of a sequence of separate tasks)
   If a procedure can be done in a sequence of two steps, with $n_1$ ways to do the first step, and $n_2$ to do the second one, then there are $n_1 n_2$ steps to do the procedure. (Generally, there are $n_1 n_2 \cdots n_k$ ways to do a procedure with $k$ steps)

2. sum rule (used when the procedure can be accomplished in more ways (independently) If a procedure can be done in a either one of two steps, with $n_1$ ways to do the first step, and $n_2$ to do the second one, then there are $n_1 + n_2$ steps to do the procedure. (Generally, there are $n_1 + n_2 + \cdots + n_k$ ways to do a procedure in $k$ independent ways)

## 5.2 The Pigeonhole Principle

1. The Pigeonhole Principle: If $k$ is a positive integer and $k + 1$ or more objects are placed into $k$ boxes, then there is at least one box containing two or more objects.

2. Generalized Pigeonhole Principle: If $n$ objects are placed into $k$ boxes, then at least one box contains at least $\lceil \frac{n}{k} \rceil$.

## 5.3 Permutations and Combinations

1. a <u>permutation</u> of a set of distinct objects is an ordered arrangement of these objects

2. an $r$-<u>permutation</u> is an ordered arrangement of $r$ elements of a set of distinct objects (Example: the 2-permutations of the elements $1, 2, 3$ are $12, 21, 13, 31, 23, 32$, so there are 6 such arrangements)

3. the number of $r$-permutations is $P(n, r) = n(n-1) \cdots (n-r+1) = \frac{n!}{(n-r)!}$

4. an $r$-<u>combination</u> of elements of a set is an unordered selection of $r$ elements from the set (Example: the 2-combinations of the elements $1, 2, 3$ are $12, 13, 23$, so there are 3 such combinations)

5. the number of $r$-combinations is $C(n, r) = \frac{n!}{r!(n-r)!}$

6. $C(n, r)$ is called the binomial coefficient, and it is commonly denoted by $\binom{n}{r}$

7. $C(n, r) = C(n, n-r)$

8. $C(n, r) = \frac{P(n,r)}{r!}$

9. common expansions of $r$-combinations:

   - $\binom{n}{0} = \binom{n}{n} = 1$ which means that there is exactly one way of choosing the empty set, and also exactly one way of choosing all the $n$ elements
   - $\binom{n}{1} = \binom{n}{n-1} = n$

   - $\binom{n}{2} = \binom{n}{n-2} = \frac{n(n-1)}{2}$

10. <u>combinatorial proof</u> (details in Section5.4): another proof technique of identities that involve counting arguments in different ways. For example:
    Prove $C(n, r) = \frac{P(n,r)}{r!}$.
    **Proof**: Let $S$ be a set of $n$ elements, and $A$ be a subset of $r$ elements of $S$. The left hand side, $C(n, r)$, counts the number of un-arranged subsets $A$. On the other hand, $P(n, r)$ counts the total number of ordered arrangements. For every subset $A$ there are exactly $r!$ ways to arrange the elements of $A$, and so when dividing $P(n, r)$ by $r!$ we obtain exactly the unordered sets of $r$ elements.

## 5.4  Binomial Coefficients

1. The Binomial Thm. gives the coefficients of the expansion $(x+y)^n$:

$$(x+y)^n = \sum_{i=0}^{n} \binom{n}{i} x^{n-i} y^i$$

$$= \binom{n}{0} x^n + \binom{n}{1} x^{n-1} y + \binom{n}{2} x^{n-2} y^2 + \ldots + \binom{n}{n-1} x y^{n-1} + \binom{n}{n} y^n$$

2. Note that Binomial Thm could be use for the expansions: $(x-y)^n = (x+(-y))^n = \sum_{i=0}^{n} \binom{n}{i} x^{n-i} (-y)^i$ or $(3x-5y)^n = ((3x)+(-5y))^n = \sum_{i=0}^{n} \binom{n}{i} (3x)^{n-i} (-5y)^i$

3. Pascal's Identity: $\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$, for $n, k \in \mathbb{N} - \{0\}, k \leq n$

4. Pascal's Triangle: gives the coefficients of $(x+y)^n$ in the $n+1$st row of the triangle. Also, it is constructed using the fact that the sum of any two consecutive binomial coefficients of the same row of the triangle is the binomial coefficient in the next row between these two coefficients (i.e. it uses Pascal's Identity)

5. The algebraic proof of the identity $C(n,r) = C(n, n-r)$ is straightforward. But there is another way, equally simple. This is called *combinatorial proof.* For our purposes, combinatorial proof is a technique by which we can prove an algebraic identity without using algebra, by finding a set whose cardinality is described by both sides of the equation. Here is a combinatorial proof that $C(n,r) = C(n, n-r)$.

   **Proof**: We can partition an $n$-set into two subsets, with respective cardinalities $r$ and $n-r$, in two ways: we can first select an $r$-combination, leaving behind its complement, which has cardinality $n-r$, or we can first take an $(n-r)$-combination, then leaving behind *its* complement, which has cardinality $r$. The number of possible outcomes is the same either way. It follows that $C(n,r) = C(n, n-r)$. ∎

   It's a remarkable method, that doesn't apply in every instance, but there are times when it is far easier to devise a combinatorial proof than an algebraic one.

6. Vandermonde's Thm $\binom{m+n}{r} = \sum_{k=0}^{r} \binom{m}{r-k} \binom{n}{k}$, for $m, n, r \in \mathbb{N}, r \leq m, n$

7. $\binom{n+1}{r+1} = \sum_{j=r}^{n} \binom{j}{r}$, for $n, r \in \mathbb{N}$

## 5.5    Generalized Permutations and Combinations

1. for generalized permutations and combinations not all the elements are distinct

2. permutations with repetitions: the number of $r$-permutations of a set of $n$ objects with repetition allowed is $\underbrace{n \cdot n \cdots n}_{r} = n^r$.

   Example: How many 4-digit pin numbers are there? Ans: $10 \cdot 10 \cdot 10 \cdot 10 = 10^4$

3. combinations with repetitions: the number of $r$-combinations of a set of $n$ objects with repetition allowed is $C(n + r - 1, r) = C(n + r - 1, n - 1)$
   Example: How many way are there to select 3 bills out of an unlimited number of $\$1, \$5, \$10$ and $\$20$ bills? Ans: Since the order they are picked doesn't matter, we have $C(n + r - 1, r) = C(3 + 4 - 1, 3) = C(6, 3) = \frac{6!}{3!3!} = 20$

4. permutations with indistinguishable objects: The number of different permutations of $n$ objects, where $n_i$ are indistinguishable objects of type $i$ with $1 \leq i \leq k$ $(n_1 + n_2 + \ldots + n_k = n)$ is

$$\binom{n}{n_1}\binom{n - n_1}{n_2} \cdots \binom{n - n_1 - \ldots n_{k-1}}{n_k} = \frac{n!}{n_1! n_2! \cdots n_k!}$$

   Example: In how many distinct ways the letters in the word "'MISSISSIPPI' can be rearranged? Ans: $\frac{11!}{1!4!4!2!}$

5. distributing objects into boxes (i.e. partitioning the objects into subsets)

6. Distributing objects into  distinguishable boxes - $\exists$ formulae for counting them

   - distinguishable objects (i.e. distributing cards to players): The number of ways to distribute objects into $k$ distinguishable boxes so that $n_i$ objects are placed into box $i$ $(1 \leq i \leq k)$ is $\binom{n}{n_1}\binom{n-n_1}{n_2} \cdots \binom{n-n_1-\ldots n_{k-1}}{n_k} = \frac{n!}{n_1! n_2! \cdots n_k!}$

   - indistinguishable objects (i.e. distributing balls into different colored bins): The number of ways to distribute objects into $k$ distinguishable boxes so that $n_i$ objects are placed into box $i$ $(1 \leq i \leq k)$ is $C(n + r - 1, r) = C(n + r - 1, n - 1)$

7. Distributing objects into indistinguishable boxes - no closed formula (see Examples 10 and 11)